The “Gamma plus two” method for generating “odd order” magic squares, the “Gamma plus two plus swap” method for generating “singly even order” magic squares, and Durer’s method for generating “doubly even order” magic squares.

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Introduction.

A magic square is an arrangement of the integers from 1 to M^2 in an MxM matrix, with each integer occurring exactly once, and such that the sum of the entries of any row, any column, or any main diagonal is the same.

This sum is usually referred to as S. It may be shown that $S = \frac{M(M^2+1)}{2}$.

The simplest magic square is the 1x1 magic square whose only entry is the number 1.

1

The next simplest is the 3x3 magic square where the sum is 15.

2	7	6
9	5	1
4	3	8

There are three types of magic squares:

1) M is an odd number (1, 3, 5, 7, etc.); M=2n+1 where n=0,1,2,3…These are referred to as “odd order” magic squares.
2) M is an even number divisible by both 2 and 4 (4, 8, 12, etc.); M=4(n+1) where n=0,1,2,3… These are referred to as “doubly even order” magic squares.
3) M is an even number divisible by 2 but not by 4 (6, 10, 14, etc.); M=2(n+3) where n=0,1,2,3.. These are referred to as “singly even order” magic squares.

Over the past four thousand years numerous methods, some relatively simple and some fairly complex, for generating magic squares of the three different types have been described. I have devised a very simple method of forming any size “odd order” magic square. I named the method “Γ + 2” (Gamma plus two) because this name describes the method. I independently developed the eight variations of this method during December of 2007. However, upon further reading about magic squares, I learned that Claude Gaspar Bachet (1581-1638) proposed a procedure similar to variation 7 of the “Gamma plus two” method. Bachet’s method, in turn, was very similar to a system written in 1693

by the mathematician Simon de la Loubere. It seems that La Loubere learned this method in Siam while he was the envoy of King Louis XIV to that country in 1687 and 1688. Further investigation showed me that in 1604 the German mathematician Johann Faulhaber published a 15x15 magic square similar to version 1 of the "Gamma plus two" method in the "Arithmeticher Cubic-cissisher Lustgarten" ("Arithmetic-Algebraic Pleasure Garden") published by Tubinger. The author of the magic square was unknown. I did not find any reference to the other six variations of the "Gamma plus two" method. I also have generalized an existing procedure, proposed by Ralph Strachey in 1918, of generating "singly even order" magic squares. I called this the "Γ + 2+Swap", or less modestly the "Brumgnach-Strachey" method. I believe that my cell swap portion of the method is innovative and has not been described before. Upon further reading on magic squares, I found that Ivan Ormsbee in April 2006 proposed a matrix addition procedure which when applied to Strachey's method gives the same results as my "Γ + 2 + Swap" method. Ormsbee used addition and subtraction of matrices to accomplish what I did with swapping.
"Doubly even order" magic squares may be easily constructed by a method described by Albrecht Durer in 1514. I will describe his method later on.

Some definitions

The MxM square is divided into M^2 number of *cells*.
Rows are numbered from the top.
Columns are numbered from the left.
The top left to bottom right main diagonal is the *left diagonal*.
The top right to bottom left main diagonal is the *right diagonal*.
The h^{th} row or column and the $(M+1-h)^{th}$ row or columns are *complementary*.
The k^{th} cell in the h^{th} row is *skewly related* to the $(M+1-k)^{th}$ cell in the $(M+1-h)^{th}$ row.
Skewly related cells are symmetrical to the center of the MxM square.
The sum of all integers in each row, each column, and the left and right diagonals is the square's constant S, and $S = \frac{M(M^2+1)}{2}$.

Short History of Magic Squares

There seems to be general agreement that magic squares were first introduced into the literature through a Chinese legend involving a turtle with a 3x3 magic square on its back. An ancient Chinese book called *Lo Shu* relates the story of how this magic turtle saved the day. Other Chinese literature claims that the *Lo Shu* magic square was invented as early as 2800 B.C. by Fuh-Hi, the mythical founder of the Chinese civilization. Magic squares appeared in Greek writings at about 1300 B.C. In the ninth century Arabian astrologers used magic squares in making up horoscopes. It seems to have been in Arabia that simple rules for producing magic squares were first devised. Indian writings of the eleventh century also mention magic squares show that magic squares were used for perfume making and medical remedies. Around 1300 A.D. the Greek mathematician Manual Moschopoulos wrote a book about the mathematics of the Arab mathematician Al-Buni through which he introduced magic squares to the West. The German physician and theologian Cornelius Agrippa compiled and used magic squares in astrology in early 1500. Claude Gaspar Bachet (1581-1638), Simon de la Loubere (1642-1729), John

Faulhaber (1604), and Albert Durer (1514) are among the noteworthy people that worked on magic squares in Europe. In the US, Benjamin Franklin was a magic square enthusiast.
The reference section has a number of some very good reference books and articles on the history of magic squares.

“Odd order” magic squares.

Before I describe the “Gamma plus two” method of generating “odd order” magic squares, it helps if we realize that magic squares are actually a two dimensional representation of the surface of a doughnut; making the upper edge continuous with the lower edge, and the right edge continuous with the left edge.

Here is how variation 1 of the “Γ + 2” method works.

1) Place the number 1 in the cell immediately to the right of the center cell. Go up one cell and then go one cell to the right and place the number 2. “Up and to the right” is how the upper case Greek Gamma (Γ) is written, ergo the first part of the name. Continue this “one up and one to the right” process.

2) Since, as we said before, the square is actually a two dimensional representation of the surface of a doughnut, when you reach the upper edge of the square, continue in the same column at the lower edge; and when you reach the right edge of the square, continue in the same row on the left edge.

3) After inserting M numbers, the “one cell up and one cell to the to the right” motion will land you on a cell that is already occupied.

4) When this happens, go back to the last entry, move two cells to the right, and insert the next number. This is where I get the “+2” part of the name.

5) Continue inserting numbers in cells “one cell up and one cell to one the right” until, after M numbers, you find the next cell that is occupied. At this point you are back to step 3.

6) Continue the process until every cell of the square is filled.

You have just created a magic square where the sum of the numbers in each horizontal line, each vertical line, and each main diagonal has the value of S.

Here is an example for a 7x7 magic square generated by the “Γ + 2” method. The sum for a 7x7 magic square is 175. In the example each cell is identified by the letter “c” and a subscript containing the row number and the column number separated by a comma. The center cell is $c_{4,4}$.

	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7
Row 1	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{1,5}$	$c_{1,6}$	$c_{1,7}$
Row 2	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$	$c_{2,4}$	$c_{2,5}$	$c_{2,6}$	$c_{2,7}$
Row 3	$c_{3,1}$	$c_{3,2}$	$c_{3,3}$	$c_{3,4}$	$c_{3,5}$	$c_{3,6}$	$c_{3,7}$
Row 4	$c_{4,1}$	$c_{4,2}$	$c_{4,3}$	$c_{4,4}$	$c_{4,5}$	$c_{4,6}$	$c_{4,7}$
Row 5	$c_{5,1}$	$c_{5,2}$	$c_{5,3}$	$c_{5,4}$	$c_{5,5}$	$c_{5,6}$	$c_{5,7}$
Row 6	$c_{6,1}$	$c_{6,2}$	$c_{6,3}$	$c_{6,4}$	$c_{6,5}$	$c_{6,6}$	$c_{6,7}$
Row 7	$c_{7,1}$	$c_{7,2}$	$c_{7,3}$	$c_{7,4}$	$c_{7,5}$	$c_{7,6}$	$c_{7,7}$

	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Sum 175
Row 1	4	29	12	37	20	45	28	175
Row 2	35	11	36	19	44	27	3	175
Row 3	10	42	18	43	26	2	34	175
Row 4	41	17	49	25	1	33	9	175
Row 5	16	48	24	7	32	8	40	175
Row 6	47	23	6	31	14	39	15	175
Row 7	22	5	30	13	38	21	46	175
Sum	175	175	175	175	175	175	175	175

Start by identifying the cell immediately to the right of center which is cell $c_{4,5}$. In this cell insert the number 1.

Go up one cell and one cell to the right. This lands you in cell $c_{3,6}$, where you place the number 2.

Go up one cell and one cell to the right. This lands you in cell $c_{2,7}$, where you place the number 3.

Go up one cell, but when you try to go right you meet the right edge of the square. When this happens jump to the leftmost cell of the square in this row which is cell $c_{1,1}$. So the number 4 is placed in cell $c_{1,1}$.

When you try to go up you meet the upper edge of the square. When this happens jump to the corresponding cell on the lower edge of the square ($c_{7,1}$); then go one cell to the right. This lands you in cell $c_{7,2}$ where you write the number 5.

Up and to the right lands you in cell $c_{6,3}$ where you write the number 6.

Up and to the right lands you in cell $c_{5,4}$ where you write the number 7.

The next "up and to the right" motion lands you in cell $c_{4,5}$ which is occupied. When this happens go back to the last entry and move two cells to the right (for this example, this will occur every 7 entries; for a MxM square this occurs every M entries). This lands you in cell $c_{5,6}$ where you write the number 8.

The process continues until all 25 numbers are entered into the square. The sum of the numbers in each row, in each column, and in each of the two main diagonals is 175. The following table summarized the placement of each number.

Number	Cell Row,Column	Next move	Number	Cell Row,Column	Next move
1	Start at 4,5	↑ →	26	3,5	↑ →
2	3,6	↑ →	27	2,6	↑ →
3	2,7	↑ →	28	1,7	→ →
4	1,1	↑ →	29	1,2	↑ →
5	7,2	↑ →	30	7,3	↑ →
6	6,3	↑ →	31	6,4	↑ →
7	5,4	→ →	32	5,5	↑ →
8	5,6	↑ →	33	4,6	↑ →
9	4,7	↑ →	34	3,7	↑ →
10	3,1	↑ →	35	2,1	→ →
11	2,2	↑ →	36	2,3	↑ →
12	1,3	↑ →	37	1,4	↑ →
13	7,4	↑ →	38	7,5	↑ →
14	6,5	→ →	39	6,6	↑ →
15	6,7	↑ →	40	5,7	↑ →
16	5,1	↑ →	41	4,1	↑ →
17	4,2	↑ →	42	3,2	→ →
18	3,3	↑ →	43	3,4	↑ →
19	2,4	↑ →	44	2,5	↑ →
20	1,5	↑ →	45	1,6	↑ →
21	7,6	→ →	46	7,7	↑ →
22	7,1	↑ →	47	6,1	↑ →
23	6,2	↑ →	48	5,2	↑ →
24	5,3	↑ →	49	4,3	End
25	4,4	↑ →			

The "Γ + 2" (Gamma plus two) method can be used to generate any MxM magic square where M is an odd number.

There are eight variations to the "Gamma plus two" method for generating odd order magic squares. The one described above (variation 1, Up Right Right Right) starts by writing the number 1 immediately to the right of the center cell. The method then proceeds by moving one cell up and one cell to the right. If the cell is empty, write the next number; if the cell is occupied, go back to the last cell, move two cells to the right and write the next number. The second variation (variation 2, Down Right Right Right) consists of writing the number 1 immediately to the right of the center cell but then moving one cell down and one cell to the right. The rest of the method is the same.

The third variation (variation 3) starts by writing the number 1 in the cell immediately below the center cell. The method then proceeds by moving one cell to the right and cell down. If the cell is empty, write the next number; if the cell is occupied, go back to the last cell, move two cells down and write the next number. The fourth variation (variation 4) is the same except the right movement is replaced by a left movement.

The fifth and sixth variations (variations 5 and 6) begin by placing the number 1 immediately to the left of the center cell. Variation 5 then goes up one cell and goes one

cell to the left while variation 6 goes down one cell and one cell to the left. If the cell is empty, write the next number; if the cell is occupied, go back, go two cells to the left and write the next number.
The seventh and eighth variations start by writing the integer 1 immediately above the center cell. Variation 7 then goes one cell to the right and one cell up while variation 8 goes one cell to the left and one cell up. If the cell is empty, write the next integer; if the cell is occupied, go back, go two cells up, and write the next integer.
The following table summarized the eight variations for generating odd order magic squares using the "Gamma plus two" method.

Brumgnach's "Γ+2" method variations for "odd order" magic squares.				
Variation of Method	Start	Next Move If next cell is empty	Next Move If next cell is occupied	
1 (UR RR)	Right of Center	↑→	→→	
2 (DR RR)	Right of Center	↓→	→→	
3 (RD DD)	Below Center	→↓	↓↓	
4 (LD DD)	Below Center	←↓	↓↓	
5 (UL LL)	Left of Center	↑←	←←	
6 (DL LL)	Left of Center	↓←	←←	
7 (RU UU)	Above Center	→↑	↑↑	
8 (LU UU)	Above Center	→↓	↑↑	

<u>"Singly even order" magic squares.</u>

Ralph Strachey has shown (as communicated in a letter to W.W. Rouse Ball in 1918) that a "singly even order" MxM magic square with integers 1 through M^2 can be constructed by combining four "odd order" NxN magic squares where $N = \frac{M}{2}$ and are placed in the following configuration:

A	C
D	B

Where A is a NxN magic square with integers ranging from 1 to N^2,
B is a NxN magic square with integers ranging from to (N^2+1) to $2N^2$,
C is a NxN magic square with integers ranging from to $(2N^2+1)$ to $3N^2$, and
D is a NxN magic square with integers ranging from to $(3N^2+1)$ to $4N^2$.

By swapping ($\frac{M^2}{4} - M$) integers between the upper half and the lower half of the ACDB structure, a magic square can be generated. According to the literature, magic squares of "singly even order" are generally the most difficult to construct, because their construction is largely empirical. The literature further states that the exact pattern of swapping seems to be arbitrary.

I believe that I have found a general, orderly, and non- arbitrary method of swapping. I named it the "Γ + 2+Swap" method, or less modestly the "Brumgnach-Strachey" method.

To begin with, generate the four magic squares A, B, C, and D using any of the 8 variations of the "Γ + 2" method, and place them in the ACDB configuration shown above. The following observations can then be made:

1) The sum of all the cells each column of the resulting MxM square is equal to the square's constant "S".

2) The sum of all the cells in each row in the top half of the MxM square (sub-squares A and C) is the same, but is always $\frac{M^3}{8}$ less than the square's constant "S". This can be written as $TS = \frac{3M^3 + 4M}{8}$.

3) The sum of all the cells in each row in the bottom half of the square (sub-squares D and B) is the same, but is always $\frac{M^3}{8}$ more than the square's constant "S". This can be written as $BS = \frac{5M^3 + 4M}{8}$.

4) We can call the constant difference between the top and bottom halves of the MxM square the "Top Bottom Difference Constant", and abbreviate it as TBDC. This can be written as $TBDC = \frac{M^3}{4}$.

5) Let the sum of the leftmost $\frac{M-2}{4}$ cells and the rightmost $\frac{M-6}{4}$ cells in the top row of the AC group be represented by TPS (Top Partial Sum).

6) Let the sum of the leftmost $\frac{M-2}{4}$ cells and the rightmost $\frac{M-6}{4}$ cells in the top row of the DB group be represented by BPS (Bottom Partial Sum).

7) The difference between BPS and TPS turns out to be the "Top Bottom Difference Constant" or TBDC. This can be written as

BPS-TPS=TBDC.

8) Therefore by swapping the leftmost $\frac{M-2}{4}$ and the rightmost $\frac{M-6}{4}$ cells of the top row between the top half and the bottom half of the ACDB structure, the sum of the cells in the top row of the AC group is increased by TBDC while the sum of the top row of the DB group is decreased by TBDC.

9) The sum of the integers in the resulting rows in both the AC and the DB groups becomes the square's constant "S".

10) This process of swapping between the top half and the bottom half of the ACDB structure is repeated for each row except the middle row.

11) In the middle row the number of cells to be swapped between A and D is also $\frac{M-2}{4}$, but the swap begins in the second column; the swap between C and B remains the rightmost $\frac{M-6}{4}$ cells. This also adjusts the sum of the main diagonals.

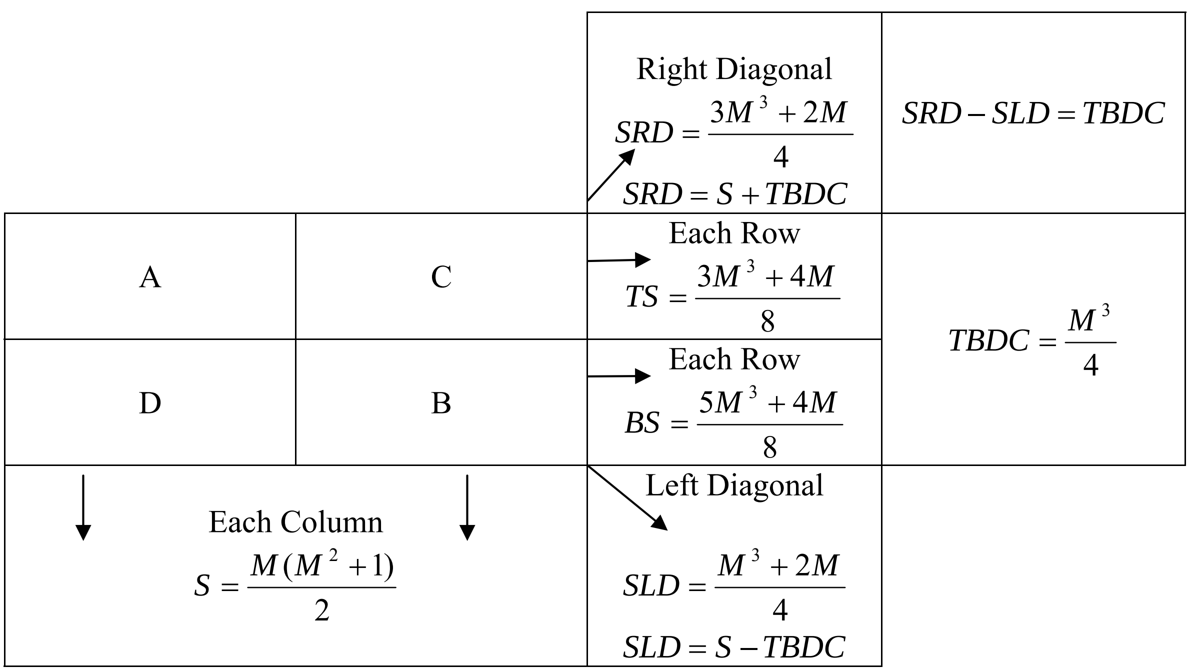
12) The main diagonal from the upper left corner to the lower right corner is called the left diagonal. The main diagonal from the upper right corner to the lower left corner is called the right diagonal. Before the swapping, the sum of the integers in the left diagonal, which can be called SLD, is $SLD = \frac{M^3+2M}{4}$. SLD is always the “Top Bottom Difference Constant” less that the square’s sum. This can be written as $SLD = S - TBDC$.

13) The sum of the integers in the right diagonal, which can be called SRD, and is $SRD = \frac{3M^3+2M}{4}$. SRD is always TBDC more than the square’s sum. This can be written as $SRD = S + TBDC$.

14) The difference between SRD and SLD is always twice the TBDC.

15) By making the swaps described above the value of SLD is increased by TBDC and the value of SRD is decreased by TBDC canceling the difference between them and making the value of both equal to S.

The following diagram summarizes some of these observations.



The following table summarizes these rules.

Rules for cell swapping for the "Γ + 2+Swap" method for "singly even order" magic squares.				
	Range of columns for cells to be swapped between top and bottom half of ACDB structure between A and D.		Range of columns for cells to be swapped between top and bottom half of ACDB structure between C and B.	
	Start of range	End of range	Start of range	End of range
All rows except middle. *OtherSwaps*	1	$\frac{M-2}{4}$	$\frac{3M+10}{4}$	M
Middle row. *MiddleSwaps* If (Start of range)>M there is no swap.	2	$\frac{M+2}{4}$	$\frac{3M+10}{4}$	M

By swapping the appropriate cells, a magic square of MxM dimensions is generated.

Alternately, the same result can be obtained by swapping columns. There are always $\frac{M+2}{4}$ columns involved in the swaps between A and D and the swaps always take place on the left side. In the leftmost column, all corresponding cells, except the middle one, are always swapped between A and D. Only the middle cell gets swapped between A and D in the $\frac{M+2}{4}$ column. In the AD group, the in-between columns get swapped in their entirety. In all cases, always swap all the corresponding cells in the rightmost $\frac{M-6}{4}$ columns between C and B in their entirety. The following table gives a summary.

Rules for swaps by columns.	
Swaps between A and D	Swaps between C and B
Leftmost column: swap all cells except the middle.	Always swap the **rightmost** $\frac{M-6}{4}$ **columns** entirely.
Column number $\frac{M+2}{4}$**:** swap only the middle cells.	
In-between columns get swapped entirely.	

I believe that this swapping pattern is my innovation and has not been described before in the literature of magic squares. The following table summarizes the number of columns to be swapped between A and D, and C and B in the "Brumgnach-Strachey" method.

Number of Columns involved in swap		
M	# of Columns to Swap Between A nd D	# of Columns to Swap Between C and B
6	2	0
10	3	1
14	4	2
18	5	3
22	6	4
26	7	5
30	8	6
…	…	…
M	$\frac{M+2}{4}$	$\frac{M-6}{4}$

Following is an example with M=6 and N=3 using variation 7 of the "Γ + 2" method for generating the odd order 3x3 sub-squares.

						165
8	1	6	26	19	24	84
3	5	7	21	23	25	84
4	9	2	22	27	20	84
35	28	33	17	10	15	138
30	32	34	12	14	16	138
31	36	29	13	18	11	138
111	111	111	111	111	111	57

Here are some observations about the 6x6 square. The sum of the integers in each row of the top half of the square is 84

($TS = \frac{3M^3 + 4M}{8} = \frac{3(6^3) + 4(6)}{8} = \frac{3(216) + 4(6)}{8} = \frac{648 + 24}{8} = \frac{672}{8} = 84$).

The sum of the integers in each row of the bottom half of the square is 138

($BS = \frac{5M^3 + 4M}{8} = \frac{5(6^3) + 4(6)}{8} = \frac{5(216) + 4(6)}{8} = \frac{1104 + 24}{8} = \frac{1104}{8} = 138$).

The sum of the integers in the left diagonal is 57

($SLD = \frac{M^3 + 2M}{4} = \frac{6^3 + 2(6)}{4} = \frac{216 + 12}{4} = \frac{228}{4} = 57$). This is 54 less than the square's sum which is 111. Notice that since TBDC is 54 ($TBDC = \frac{M^3}{4} = \frac{6^3}{4} = \frac{216}{4} = 54$) and SLD is also 54, then SLD=S-TBDC.

The sum of the integers in the right diagonal is 165

($SRD = \frac{3M^3 + 2M}{4} = \frac{3(6^3) + 2(6)}{4} = \frac{3(216) + 12}{4} = \frac{648 + 12}{4} = \frac{660}{4} = 165$). This is 54 more than the square's sum which is 111. Notice that since TBC is 54 and SRD is 165, then SRD=S+TBDC.

In order to balance all these differences, 3 cells ($\frac{M^2}{4} - M = \frac{6^2}{4} - 6 = \frac{36}{4} - 6 = 9 - 6 = 3$) have to be swapped between the top and the bottom half of the 6x6 square. The cells to be swapped are determined by the "Γ + 2+Swap" method.

Since M=6, the range of cells to be swapped in all the rows except the center row between A and D begins at column1 and ends at column 1:

$$ADOtherSwaps = \frac{M - 2}{4} = \frac{6 - 2}{4} = \frac{4}{4} = 1$$

The range of cells to be swapped in all the rows except in the middle one between C and B ends at 6 and starts at column:

$$CBOtherSswaps = \frac{3M + 10}{4} = \frac{3(6) + 10}{4} = \frac{18 + 10}{4} = \frac{28}{4} = 7.$$

Since this number is bigger than 6, and the start column number is bigger than the end column number, there is no swap between B and C.

The range of cell swaps in the middle row between A and D begins at column 2 and ends at column 2:

$$ADMiddleSwaps = \frac{M + 2}{4} = \frac{6 + 2}{4} = \frac{8}{4} = 2.$$

As before, there is no swap between B and C.

Alternately, since $\frac{M + 2}{4} = \frac{6 + 2}{4} = \frac{8}{4} = 2$, there are 2 columns involved in the AD swap. All the cells, except the middle, get swapped in the leftmost column, and only the middle cells get swapped in the next column. Since $\frac{M - 6}{4} = \frac{6 - 6}{4} = 0$, there are no swaps between C and B.

So the leftmost column in the A sub-square gets swapped with the leftmost column in the D sub-square, except for the center cell. The 8 and 4 get swapped with the 35 and 31. The 30 and the 3 do not get swapped. In the next column, the 5 and the 32 get swapped. There is no swap between C and B. The resulting 6x6 magic square has a sum of 111.

						111
35	1	6	26	19	24	111
3	32	7	21	23	25	111
31	9	2	22	27	20	111
8	28	33	17	10	15	111
30	5	34	12	14	16	111
4	36	29	13	18	11	111
111	111	111	111	111	111	111

Following is an example with M=14 and N=7 using variation 7 of the "Γ + 2" method for generating the odd order 7x7 sub-squares.

The following observations can be made about the 14x14 square.

$$S = \frac{M(M^2+1)}{2} = \frac{14(14^2+1)}{2} = \frac{14(196+1)}{2} = \frac{14(197)}{2} = 7(197) = 1379$$

$$TS = \frac{3M^3+4M}{8} = \frac{3(14^3)+4(14)}{8} = \frac{3(2744)+4(14)}{8} = \frac{8232+56}{8} = \frac{8288}{8} = 1036$$

$$BS = \frac{5M^3+4M}{8} = \frac{5(14^3)+4(14)}{8} = \frac{5(2744)+4(14)}{8} = \frac{13720+56}{8} = \frac{13776}{8} = 1722$$

$$SLD = \frac{M^3+2M}{4} = \frac{14^3+2(14)}{4} = \frac{2744+28}{4} = \frac{2722}{4} = 693$$

$$SRD = \frac{3M^3+2M}{4} = \frac{3(14^3)+2(14)}{4} = \frac{3(2744)+28}{4} = \frac{8232+28}{4} = \frac{8260}{4} = 2065$$

$$TBDC = \frac{M^3}{4} = \frac{14^3}{4} = \frac{2744}{4} = 686$$

$$SLD = S - TBDC = 1379 - 686) = 693$$

$$SRD = S + TBDC = 1379 + 686 = 2065$$

														2065
46	15	40	9	34	3	28	144	113	138	107	132	101	126	1036
21	39	8	33	2	27	45	119	137	106	131	100	125	143	1036
38	14	32	1	26	44	20	136	112	130	99	124	142	118	1036
13	31	7	25	43	19	37	111	129	105	123	141	117	135	1036
30	6	24	49	18	36	12	128	104	122	147	116	134	110	1036
5	23	48	17	42	11	29	103	121	146	115	140	109	127	1036
22	47	16	41	10	35	4	120	145	114	139	108	133	102	1036
193	162	187	156	181	150	175	95	64	89	58	83	52	77	1722
168	186	155	180	149	174	192	70	88	57	82	51	76	94	1722
185	161	179	148	173	191	167	87	63	81	50	75	93	69	1722
160	178	154	172	190	166	184	62	80	56	74	92	68	86	1722
177	153	171	196	165	183	159	79	55	73	98	67	85	61	1722
152	170	195	164	189	158	176	54	72	97	66	91	60	78	1722
169	194	163	188	157	182	151	71	96	65	90	59	84	53	1722
1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	693

Since M=14 the cells to be swapped between A and D in all the rows except the middle range from column1 through column 3:

$$ADOtherSwaps = \frac{M-2}{4} = \frac{14-2}{4} = \frac{12}{4} = 3\,.$$

The range of cells to be swapped between C and B in all the rows except the middle ends at column M (or 14) and starts at column 13:

$$CBOtherSwaps = \frac{3M+10}{4} = \frac{42+10}{4} = \frac{52}{4} = 13\,.$$

The cells to be swapped between A and D in the middle row, range from column 2 through column 4:

$$ADMiddleSwaps = \frac{M+2}{4} = \frac{14+2}{4} = \frac{16}{4} = 4\,.$$

Alternately, since $\frac{M+2}{4} = \frac{14+2}{4} = \frac{16}{4} = 4$, there are 4 columns involved in the AD swap. In the leftmost column swap all the corresponding cells except the middle. The next two columns swap in their entirety. In the fourth column, only swap the middle cells. Since $\frac{M-6}{4} = \frac{14-6}{4} = \frac{8}{4} = 2$, swap the rightmost 2 columns in their entirety between C and B.

All the cells (except the middle) of the leftmost column of sub-square A (46, 21, 38, 30, 5, 22) gets swapped with the corresponding cells in the leftmost column of sub-square B (193, 168, 185, 177, 152, 169). The second column of A gets entirely swapped with the second column of D, the third column of A gets entirely swapped with the third column of D, and the middle cell of the fourth column of A gets swapped with the middle cell of the fourth column in D. The rightmost two columns of C (cells in columns 13 and 14) get swapped with the rightmost two columns of B in their entirety.

The following diagram is the resulting 14x14 magic square with 1,379 as the sum.

														1379
193	162	187	9	34	3	28	144	113	138	107	132	52	77	1379
168	186	155	33	2	27	45	119	137	106	131	100	76	94	1379
185	161	179	1	26	44	20	136	112	130	99	124	93	69	1379
13	178	154	172	43	19	37	111	129	105	123	141	68	86	1379
177	153	171	49	18	36	12	128	104	122	147	116	85	61	1379
152	170	195	17	42	11	29	103	121	146	115	140	60	78	1379
169	194	163	41	10	35	4	120	145	114	139	108	84	53	1379
46	15	40	156	181	150	175	95	64	89	58	83	101	126	1379
21	39	8	180	149	174	192	70	88	57	82	51	125	143	1379
38	14	32	148	173	191	167	87	63	81	50	75	142	118	1379
160	31	7	25	190	166	184	62	80	56	74	92	117	135	1379
30	6	24	196	165	183	159	79	55	73	98	67	134	110	1379
5	23	48	164	189	158	176	54	72	97	66	91	109	127	1379
22	47	16	188	157	182	151	71	96	65	90	59	133	102	1379
1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379

The Durer method for "doubly even order" magic squares.

"Doubly even order" magic squares may be easily constructed by a method described in 1514 by the German painter, wood carver, engraver, and mathematician Albrecht Durer. First draw a square containing NxN cells. Break up the cells into 4x4 groups. Draw imaginary diagonals through each 4x4 sub-square. Write the number 1 in the upper left hand corner cell and proceed horizontally to the right with consecutive numbers, but only write the numbers in the cells that are crossed by diagonals. Continue this for each row. When you have filled in the bottom right hand cell, make believe that the number 1 is in this cell and proceed horizontally to the left writing in the vacant cells.

For a 4x4 square we have the following diagrams. For a 4x4 magic square the sum is 34. The first diagram shows the "forward" entries.

1			4
	6	7	
	10	11	
13			16

The next diagram is the finished square which includes the "reverse" entries.

				34
1	15	14	4	34
12	6	7	9	34
8	10	11	5	34
13	3	2	16	34
34	34	34	34	34

Here an example of an 8x8 magic square. The sum here is 260.

								260
1	63	62	4	5	59	58	8	260
56	10	11	53	52	14	15	49	260
48	18	19	45	44	22	23	41	260
25	39	38	28	29	35	34	32	260
33	31	30	36	37	27	26	40	260
24	42	43	21	20	46	47	17	260
16	50	51	13	12	54	55	9	260
57	7	6	60	61	3	2	64	260
260	260	260	260	260	260	260	260	260

Summary of Symbols and Formulas

M = order of the magic square

F = first number to be entered (beginning number)

GI = Gamma Increment

PI = Plus two Increment

PS = Pure Sum (if F=1, GI=1, and PI=1)

FT = First Term in Sum Equation: $FT = M(F-1)$

CT = Computational Term Used to Simplify Sum Formula: CT = 1+2+3+…+M

GIT = Term in Sum Equation due to Gamma Increment: GIT = (CT-M)[M(GI-1)]

PIT = Term in Sum Equation due to Plus Two Increment: PIT = (CT-M)(PI-1)

SET = Term in Sum Formula due to the Brumgnach-Strachey Method:

$$SET = \frac{M^2}{2}(PI - GI)$$

For doubly even order or for odd order magic squares, SET = 0

S = Magic Sum: S = PS + FT + GIT + PIT + SET

I believe that this form of the magic sum formula is innovative, especially the SET term.

I was not able to find the SET term anywhere else in the magic squares literature.

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